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THEORY OF SINGULARITIES IN HEAT TRANSFER PROBLEMS: RELAXATION-TYPE SOLUTIONS

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The possibility of the existence of asymptotically periodic pulsed solutions for hyperbolic heat transfer equations with nonlinear boundary conditions is demonstrated.

The present work has been motivated by the investigation of certain singularities arising in nonlinear boundary conditions when heat transfer is described with allowance for heat flux relaxation [1]:

$$\rho c_v \frac{\partial T}{\partial t} = - \frac{\partial Q}{\partial s},$$

$$Q + \tau \frac{\partial Q}{\partial t} = - \kappa \frac{\partial T}{\partial s}, \quad (s, t) \in \Pi = [0, l] \times \mathbf{R}^+, \quad l > 0, \tag{1}$$

with the boundary conditions

$$Q(0, t) = 0 \quad \text{and} \quad T(l, t) = \varphi(Q(l, t)) \tag{2}$$

and the initial conditions

$$Q(s, 0) = Q_0(s) \quad \text{and} \quad T(s, 0) = T_0(s). \tag{3}$$

This problem can describe the distribution of temperature in an infinite layer [2]; moreover the boundary condition on the left is insignificant and therefore it can be taken to be the same as on the right. By substitution of unknown functions and scale transformation of variables: $Q = 0.5(u + v)$, $T = 0.5z(u - v)$, $s = lx$, $t = (l(\omega)\bar{t})$, where z and ω are constants, this problem is reduced to the form [3]

$$\begin{aligned} &\frac{\partial u}{\partial \bar{t}} + \frac{\partial u}{\partial x} = - \mu (u + v), \\ &\frac{\partial v}{\partial \bar{t}} - \frac{\partial v}{\partial x} = - \mu (u + v), \quad (x, \bar{t}) \in \bar{\Pi} = [0, 1] \times \mathbf{R}^+, \end{aligned} \tag{4}$$

where μ is a certain parameter, with the boundary conditions

$$u(0, \bar{t}) = - v(0, \bar{t}) \quad \text{and} \quad - v(1, \bar{t}) = f(u(1, \bar{t})) \tag{5}$$

and the initial conditions

$$u(x, 0) = h_1(x) \quad \text{and} \quad v(x, 0) = h_2(x). \tag{6}$$

Here $f(\cdot)$ is a function assigned implicitly by the relation

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$$\frac{z}{2}(u+f)=\varphi\left(\frac{u-f}{2}\right).$$

We note that

$$\mu=\frac{l}{2}\sqrt{\left(\frac{\rho c_v}{\kappa\tau}\right)},$$

where τ is the time of heat flux relaxation, and the remaining notation has the generally accepted meaning.

At $\mu = 0$, asymptotically periodic piecewise-constant solutions witch a finite, denumerable, or nondenumerable nowhere dense set of points of discontinuity in the period, which are usually called solutions of relaxational, preturbulent, and turbulent types, respectively, are typical for problem (4)-(6) [3]; the points of discontinuity are understood to refer to the set

$$\tau_0=\left\{x,\ t:\ \lim_{\substack{t\to t_0\\ x\in[0,1]}}\ \sup\left(|u_t|,\ |v_t|\right)=\infty\right..$$

In [3–6] it is shown that in the case of sufficiently small values of μ, $\mu > 0$, the qualitative properties of the solutions are preserved, while the set of points τ_0 is somewhat shifted so that

$$\lim_{\mu\to 0}\ \sup\ \mathrm{dist}\,(\tau_\mu,\ \tau_0)=0,$$

where "dist" is the distance between the corresponding points.

In the numerical investigation of the problem at hand the "stumbling block" is the relationship from which it is necessary to find the function f. In practical problems (see below) this relationship is frequently represented by a bilinear polynomial $\Phi(u, f) = 0$, and here we encounter a situation that is the subject of investigation in the theory of singularities, namely, we consider the possible bifurcations of the equation [7, p. 45]

$$g(u,\lambda)=0,\quad g:\ R^1\times R^1|\to R^1,$$

where λ is a parameter. As follows from the implicit function theorem, the equality $g_u(u_0, \lambda_0) = 0$ is a necessary condition for the solution (u_0, λ_0) of the equation $g(u, \lambda) = 0$ to be a bifurcation point, otherwise a single $u \in R^1$ will correspond to each $\lambda \in R^1$.

Let us consider the function

$$g(u,\lambda)=u^2-\lambda u.$$

Here $u = 0$ is the solution of the equation $g = 0$ for any λ. Moreover, since $(dg)_{0,\lambda} = -\lambda$, at $\lambda \neq 0$ the implicit function theorem guarantees that $u = 0$ is the unique solution of the equation $g(u, \lambda) = 0$ near $u = 0$. However, an analysis of the set $\{u, \lambda: g(u, \lambda) = 0\}$ shows that the neighborhood of zero, in which this theorem is valid, tends to a point when $\lambda \to 0$ [7], i.e., the point (0, 0) is a bifurcation point, that is, branching of solutions occurs.

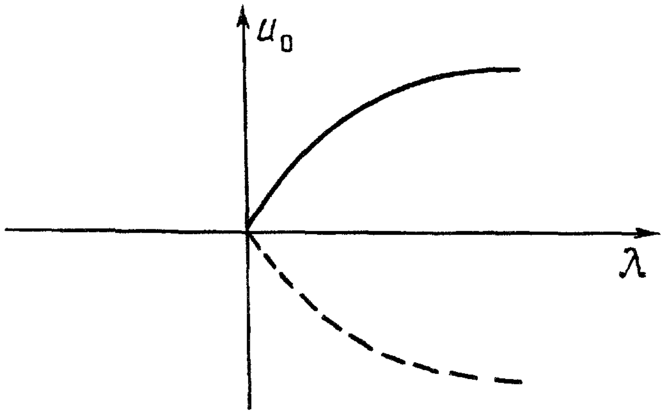
Next, we consider the case

$$L(u,\lambda)=-u^2+\lambda$$

and we assume that we must investigate the asymptotic behavior of the solutions of the equation

$$u_t=-u^2+\lambda.$$

It is not hard to show [7, p. 88] that stationary branches have the form depicted in Fig. 1, with the point $u_0 = \sqrt{\lambda}$ being stable and the point $u_0 = -\sqrt{\lambda}$ being unstable when $\lambda > 0$; when $\lambda < 0$, there are no singular points.



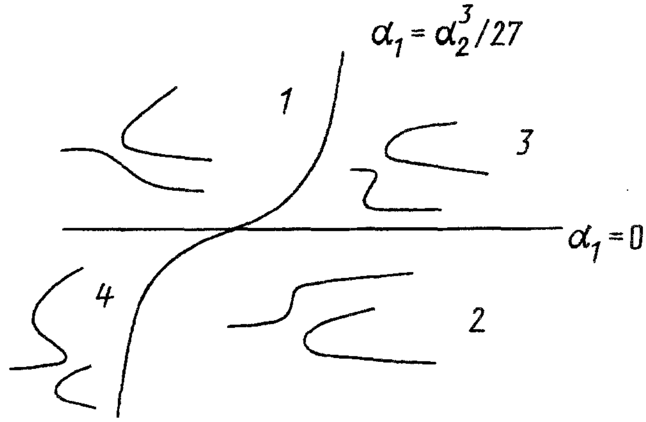


Fig. 1. "Forked" bifurcation.

Fig. 2. Classification of a boundary condition by "typical" parameters.

The first example is cited to demonstrate the way in which the theory of singularities "operates," and the second example is typical for applications. We also note that in applications the theory of singularities is used only for investigating ordinary differential equations or partial differential equations that (for reasons of their own) reduce to ordinary equations: in this case it is often overlooked that the theory is of value "by itself," being a branch of algebraic topology that is concerned with problems of the construction of surfaces for systems of equations.

It turns out that in the boundary-value problem (1)-(3) the methods of the theory of singularities can be applied "directly." In fact, suppose that on the boundary

$$Q = q_{\text{inc}} - \varepsilon(T)\, T^k, \quad k \geq 4,$$

where q_{inc} is an incident radiation flux; $\varepsilon(\cdot)$ is a certain function. Usually, the form of $\varepsilon(\cdot)$ is unknown, and one has to find it experimentally; therefore we assume

$$\varepsilon(T)\, T^k = a_0 + a_1 T + a_2 T^2 + \ldots,$$

where a_0, a_1, a_2 are unknown coefficients. Then, according to the above-indicated replacement of variables

$$\frac{1}{2}(u+v) = q_{\text{inc}} - \varepsilon\left(\frac{z}{2}(u-v)\right)\left(z\frac{u-v}{2}\right)^k, \quad k \geq 4,$$

$$\frac{1}{2}(u+v) = q_{\text{inc}} - a_0 - a_1\frac{z}{2}(u-v) - a_2\left(\frac{z}{2}\right)^2(u-v)^2 + \ldots,$$

$$u + v = -A_0 - A_1(u-v) - A_2(u-v)^2 - \ldots,$$

where

$$-A_0 = 2q_{\text{inc}} - 2a_0; \quad A_1 = a_1 z; \quad A_2 = a_2\frac{z^2}{2};$$

$$\Phi \overset{\text{def}}{=} (u+v) + A_0 + A_1(u-v) + A_2(u-v)^2 + \ldots$$

We denote $\hat{\Phi} = \Phi - A_0$, consider the image $\hat{\Phi} = (u+v) + A_1(u-v) + A_2(u-v)^2 + \ldots$, and show that the relation $\Phi(u, v) = 0$ $(v = \lambda)$ is equivalent to a "forked" bifurcation (Fig. 1). As is known [7], the singularity at the point (0, 0) is determined by the equations

$$\hat{\Phi} = \hat{\Phi}_u = 0\,; \quad \hat{\Phi}_{uu} \neq 0\,; \quad \hat{\Phi}_\lambda \neq 0$$

and is equivalent to the normal form $\pm u^2 \pm \lambda$ for a certain choice of signs, to be determined below. In fact

$$\hat{\Phi}_u (0\,,\ 0) = 1 + A_1\,, \quad A_1 = -1\,, \quad \hat{\Phi}_u (0\,,\ 0) = 0\,,$$

$$\hat{\Phi}_{uu} (0\,,\ 0) = 2A_2 \neq 0\,, \quad A_2 \neq 0\,,$$

$$\hat{\Phi}_v (0\,,\ 0) = 1 - A_1 = 2 \neq 0\,,$$

i.e., a nondegenerate change of variables should exist such that

$$\hat{\Phi} (u\,,\ v) \overset{\text{eq}}{\simeq} \pm v^2 \pm v\,.$$

Let us select $A_2 < 0$; then $\hat{\Phi}_{uv}(0, 0) < 0$, and since $\hat{\Phi}_v(0, 0) > 0$ the normal form appears as $(u, v) \simeq u^2 - v$.

Note that the universal deformation of this normal form is

$$G (u\,,\ v\,,\ \alpha) = v^3 - vu + \alpha_1 + \alpha_2 u^2\,.$$

As shown above, in order that the boundary condition that connects the temperature and the heat flux be reduced to the simplest normal form, it is sufficient to assume that

$$\varepsilon\,(T)\, T^k = A_0 - T + A_2 T^2\,, \quad k \geq 4\,,$$

where A_0 and A_2 are unknown coefficients. In this case

$$u^2 - v \simeq \Phi - A_0\,,$$

and consequently, for system (1) we can consider the boundary conditions

$$u = -v\big|_{s=0} \quad \text{and} \quad \Phi = u^2 - v + A_0\big|_{s=1} = 0\,.$$

Performing integration along the characteristics, problem (4)-(6) is reduced to the investigation of a difference equation with a continuous time [3]:

$$u\,(1\,,\ \bar{t} + 2) = u\,(0\,,\ \bar{t} + 1) = -\,v\,(0\,,\ \bar{t} + 1) =$$

$$= -\,v\,(1\,,\ \bar{t}\,) = -\,u^2\,(1\,,\ \bar{t}\,) - A_0\,,$$

i.e.,

$$u\,(1\,,\ \bar{t} + 2) = -\,u^2\,(1\,,\ \bar{t}\,) - A_0\,. \tag{7}$$

The behavior of the solutions of Eq. (7) for $\bar{t} \to \infty$ is known [3] and is characterized by the fact that for a nonempty class of initial functions prescribed over the interval [0, 2) asymptotically periodic piecewise-constant solutions exist that take values from $\mathbf{P}^+$, where $\mathbf{P}^+$ is the set of attracting fixed points of the mapping f. Note that the reduction to the difference equation is possible only at $\mu = 0$, while for $\mu > 0$ a complex integro-difference equation is obtained [3]. For example, there is a denumerable set of parameters $\mu_n = -A_{0,n} > 0$ for which solutions of (7) experience bifurcations of the doubling of periods so that the number of fluctuations on any interval $[t,\ t + 2)$ increases in exponential or power-like fashion as $t \to \infty$ [3].

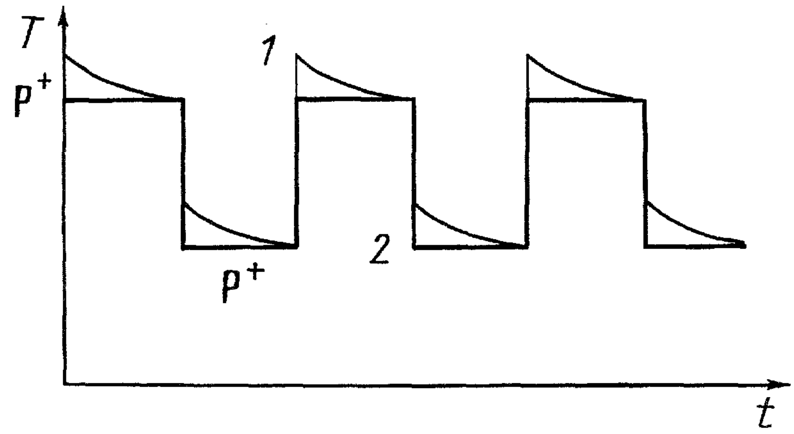


Fig. 3. Limiting distribution of temperature: 1) for $\mu > 0$; 2) for $\mu = 0$.

We note that $q_{inc} > a_0$, where a_0 plays the role of a "damper," and, finally, we note that in deformation (the form G) it is possible to assume that $\alpha_1 = A_0$, $\alpha_2 = A_2$, where $\alpha_1, \alpha_2 \in R^-$, and the corresponding deformations of the normal form G are known [7] (see Fig. 2). These deformations in the boundary condition may arise either as perturbations (when carrying out an experiment) or because of the fact that in the statement of the model problem some additional parameters were not taken into account (a detailed discussion of all the possibilities can be found in [7]). Here, similarly to the foregoing, we obtain difference equations of form (7), where the image $\tilde{\Phi}$ will be described by one of the curves 1–4. The parabola-type curves do not give new solutions; the curves in regions 2, 1 give relaxation-type fluctuations (Fig. 3), while curves 3, 4 give preturbulent-type fluctuations, i.e., the set of points of discontinuity of the limiting function is denumerable [3].

Finally, with our choice,

$$\varepsilon(T) = \frac{\alpha_0}{T^k} - \frac{1}{T^{k-1}} + \frac{\alpha_2}{T^{k-2}}, \quad k \geq 4,$$

and, moreover, it is possible to determine the set of bifurcation parameters α_0 and α_2 for which the above-indicated types of solutions are possible. If we are interested a priori in the representation

$$\varepsilon(T)\, T^k = A_0 + A_1 T + \ldots + A_k T^k + \ldots,$$

then it is possible to reduce the boundary condition to other types of normal forms whose classification is known [7] and, thus, to avoid as much as possible complex numerical calculations in studying heat transfer problems.

NOTATION

Q, heat flux; T, temperature; κ, heat conduction coefficient; s, spatial coordinate; t, time; τ, time heat flux relaxation.

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